Jet and arc spaces from a commutative algebra point of view

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CARES

11 March 2021

Outline

Topics:

- Functors of points
- Definition of jets and arcs
- Examples of jets and arcs
- Characterization of jets and arcs
- Jet/arc schemes

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Conventions:

- k is a field (you're welcome to think $\mathbf{C}$, but the characteristic doesn't matter)
- $R, S, T \in \mathbf{Alg}_k$ (you're welcome to think of finite type, i.e., $k[x_1, \ldots, x_n]/(f_1, \ldots, f_s)$)
- $m \in \mathbf{N}$
- For a category $\mathcal{C}$, $X \in \mathcal{C}$ means X is an object of $\mathcal{C}$
- I suppress the noodly hypotheses

Functors of points

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     - The "Y-valued-points" of X are $\mathrm{Hom}_{\mathbf{Top}}(Y, X)$, for any Y.

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Yoneda Lemma/Corollary. *In any category $\mathcal{C}$, $X_1 \cong X_2$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X_1) \cong \mathrm{Hom}_{\mathcal{C}}(-, X_2)$.*

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   - Does that mess with the Yoneda Lemma at all? No, work in the opposite category.
4. When we have a functor of the form $\mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \mathbf{Set}$, we say that T represents the functor.

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Do $J^m R$ and $J^\infty R$ exist?

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Representing objects are unique up to isomorphism, so $J^0 R \cong R$ for any R.

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so $a_0 b_0 = a_0 b_1 + a_1 b_0 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$.

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Therefore the map $k[x,y]/(xy) \to S[t]/t^3$ is the same as a map $k[a_0,a_1,a_2,b_0,b_1,b_2]/(a_0b_0, a_0b_1+a_1b_0, a_0b_2+a_1b_1+a_2b_0) \to S$.

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Theorem. *If $R = k[x_\alpha]/(f_\beta)$ for indices α, β, then*

$$J^m R \cong k[x_\alpha, {x_\alpha}', \ldots, {x_\alpha}^{(m)}]/(f_\beta, {f_\beta}', \ldots, {f_\beta}^{(m)})$$

and

$$J^\infty R \cong k[x_\alpha, {x_\alpha}', \ldots]/(f_\beta, {f_\beta}', \ldots).$$

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We know that for affine schemes, we can cook up jet spaces and arc spaces.

$$\operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} S, J^m \operatorname{Spec} R) \cong \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} S[t]/t^{m+1}, \operatorname{Spec} R).$$

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What about a generic (not necessarily affine) scheme?

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Proof.

$$\operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} S, J^m X) \cong \operatorname{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} S[t]/t^{m+1}, X).$$

A map $\operatorname{Spec} S[t]/t^{m+1} \to X$ factors through V if and only if $\operatorname{Spec} S \to \operatorname{Spec} S[t]/t^{m+1} \to X$ factors through V. □

Jet/arc schemes

So now we have a scheme X, an affine cover $\{U_i\}$, and a characterization of the m jets of any $V \subseteq X$; when they exist, they are $J^m V \cong {\pi_m}^{-1} V$.

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Thus we can glue all the $\{J^m U_i\}$ along these intersections to get a well-defined scheme.

Jet/arc schemes

It is then straightforward to see that this scheme, which we will call $J^m X$, is the representing object of the functor $S \mapsto \mathrm{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} S[t]/t^{m+1}, X)$.

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In other words:

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, S, J^m X) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, S[t]/t^{m+1}, X).$$

Thank you!

Lawrence Ein and Mircea Mustaţă. *Jet schemes and singularities*. Proceedings of Symposia in Pure Mathematics, p. 505-546. 2009. AMS.